Received: *January 26, 1998,* Accepted: *February 17, 1998*

Manifestly covariant actions for $D = 4$ self-dual Yang-Mills and $D = 10$ super-Yang-Mills

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Abstract: Using an infinite number of fields, we construct actions for $D = 4$ self-dual Yang-Mills with manifest Lorentz invariance and for $D = 10$ super-Yang-Mills with manifest super-Poincaré invariance. These actions are generalizations of the covariant action for the $D = 2$ chiral boson which was first studied by McClain, Wu, Yu and Wotzasek.

Keywords: Field Theories in Higher Dimensions, Duality in Gauge Field Theories, Extended Supersymmetry, Supersymmetry and Duality.

JHEP02(1998)012

Contents

1 Introduction

When a physical system has symmetries, it is useful for these symmetries to be manifest in the action. However, there is a prominent example where this is not straightforward: Lorentz invariance in self-dual systems. There are currently at least three methods available for attacking this problem.

The first method was proposed by Siegel [1] and introduces a Lagrange multiplier for the square of the self-duality constraint. This method has been most successful in the treatment of chiral scalars, for which all anomalies can be cancelled by introducing an auxiliary non-dynamical sector [2]. The resulting theory is a conformal field theory coupled to *two* world-sheet metrics, with a conformal invariance associated with each, and has the correct spectrum. However, the partition function depends on the moduli for *both* metrics and must have modular invariance for both the moduli of the original metric and for the Siegel multiplier field [3]. Nevertheless, a consistent quantum theory emerges, and at least in some cases, integrating first over the extra moduli gives the desired result.

The second method (MWYW) was developed by McClain, Wu, Yu [4] and by Wotzasek [5] for the $D = 2$ chiral boson. The chirality condition is a second-class constraint, and after introducing new fields, can be transformed into a first-class constraint. However, these new fields satisfy second-class constraints, and one has to continue the procedure ad infinitum. The final action therefore contains an infinite number of fields.

This MWYW method was later used to construct $D = 4$ Maxwell [6] and super-Maxwell [7] actions with manifest electro-magnetic duality, and to construct covariant actions in $D = 4p+2$ for $2p$-form gauge fields with self-dual field strengths [8, 9]. There is such a self-dual 4-form gauge field in the spectrum of the Type IIB superstring, and it was shown in [10] that superstring field theory utilizes the MWYW method for constructing its action.

The third method (PST) for constructing covariant actions for self-dual systems was developed by Pasti, Sorokin and Tonin [11, 12]. Starting from an action without manifest covariance, the PST method introduces a new harmonic-like field which allows the action to be written in a manifestly covariant form. This method has the advantage over the MWYW method of using a finite number of fields, but it has the disadvantage that the resulting action involves inverse powers of the harmonic-like field.

In this paper, we shall use the MWYW method to construct manifestly covariant actions for $D = 4$ self-dual Yang-Mills and $D = 10$ super-Yang-Mills with arbitrary gauge group. At the present time, we do not know how to construct analogous actions using the PST method. Note that it was shown in [11] how to obtain the MWYW action from the PST action, but it does not seem straightforward to obtain the PST action from the MWYW action.

In the second section of this paper, we shall begin by reviewing the MWYW method for the $D = 2$ chiral boson. As pointed out by Nekrassov [13], this $D = 2$ action involving an infinite number of fields can be understood as a discretized version of the $D = 3$ Chern-Simons action for an abelian one-form. We will explicitly construct this $D = 3$ Chern-Simons action and show that it is at level $\frac{1}{2}$ for a chiral boson at free-fermion radius. It is interesting to note that a level $\frac{1}{2}$ Chern-Simons action was recently used in [14] for defining the partition function for a chiral scalar.

In the third section, we use the MWYW methed to construct a $D = 4$ self-dual Yang-Mills action with manifest Lorentz-invariance, and in the fourth section, we use the MWYW method to constuct a $D = 10$ super-Yang-Mills action with manifest super-Poincaré invariance. Manifestly covariant actions for these two systems have never previously been constructed. Finally, in the fifth section, we make some comments on actions with an infinite number of fields.

2 MWYW actions for $D = 2$ chiral boson

2.1 Hamiltonian formalism

In their original papers, McClain, Wu, Yu [4] and Wotzasek [5] used the Hamiltonian formalism to describe the $D = 2$ chiral boson. Although this formalism is not manifestly Lorentz-covariant, it is useful for the analysis of physical states.

For a $D = 2$ non-chiral boson described by canonical variables $\phi(x)$ and its momentum $\Pi(x)$, the free action is $\mathcal{S} = -\mathcal{H} + \int d^2x \Pi\partial_0\phi$ where $\mathcal{H} = \frac{1}{2}\int d^2x(\Pi^2 + (\partial_1\phi)^2)$. The chirality condition

$$\Pi - \partial_1\phi = 0$$

is a second-class constraint which makes it difficult to implement using an action principle. MWYW convert it to the first-class constraint

$$\Pi - \partial_1\phi - \Pi_{(1)} - \partial_1\phi_{(1)} = 0\,,$$

where $\phi_{(1)}(x)$ and $\Pi_{(1)}(x)$ are canonical variables for a second scalar boson. In order to describe a single chiral boson, one needs to impose in addition $\Pi_{(1)} + \partial_1\phi_{(1)} = 0$. This in turn is a second-class constraint, which can itself be converted to a first class constraint by repeating the procedure and introducing yet another scalar boson described by the canonical variables $\phi_{(2)}(x)$ and $\Pi_{(2)}(x)$. This continues ad infinitum to produce the action

$$\mathcal{S} = -\mathcal{H} + \sum_{n=0}^{\infty}\int d^2x \Pi_{(n)}\partial_0\phi_{(n)}\,, \tag{2.1}$$

where

$$\mathcal{H} = \sum_{n=0}^{\infty}\int d^2x[\frac{1}{2}(-1)^n(\Pi^2_{(n)} + (\partial_1\phi_{(n)})^2) + \lambda_{(n)}T_{(n)}]\,,$$
$$T_{(n)} = \Pi_{(n)} - \partial_1\phi_{(n)} - \Pi_{(n+1)} - \partial_1\phi_{(n+1)}\,,$$

$\lambda_{(n)}$ are the Lagrange multipliers for the first-class constraints $T_{(n)} = 0$, and $\{\phi_{(0)}, \Pi_{(0)}\}$ are the canonical variables for the original scalar.

To prove that this action describes a single chiral scalar, one uses the fact that

$$\left[\, T_{(m)}(x_0, x_1)\ ,\ \phi_{(n+1)}(x_0, x_1') + \int^{x_1'} dy\ \Pi_{(n+1)}(x_0, y)\,\right] = -\delta_{mn}\delta(x_1 - x_1')$$

to gauge $\phi_{(n)}(x_0, x_1) = -\int^{x_1} dy\Pi_{(n)}(x_0, y)$ for all $n > 0$. Together with the $T_{(n)}$ constraints, this implies the desired conditions:

$$\Pi_{(0)} = \partial_1\phi_{(0)}, \quad \phi_{(n)} = \Pi_{(n)} = 0 \ \ for\ n > 0\,.$$

Although one might worry about determining the physical spectrum when there are an infinite number of fields and gauge-invariances, reference [4] confirms this result by performing a careful analysis of the BRST cohomology using the $OSp(1,1)$ method.

2.2 Lagrangian formalism

As shown in [9], the manifestly covariant form of (2.1) is obtained by solving the equations of motion for $\Pi_{(n)}$ which produces:

$$\begin{aligned}\mathcal{S} \;=\; & \frac{1}{4\pi}\sum_{n=0}^{\infty}\int d^2x[\frac{1}{2}(-1)^n\partial_+\phi_{(n)}\partial_-\phi_{(n)} + A^-_{(2n)}(\partial_-\phi_{(2n)} + \partial_-\phi_{(2n+1)}) - \\ & -A^+_{(2n+1)}(\partial_+\phi_{(2n+1)} + \partial_+\phi_{(2n+2)}) + (A^-_{(2n)} - A^-_{(2n+2)})A^+_{(2n+1)}],\end{aligned} \tag{2.2}$$

where $A^-_{(2n)}$ and $A^+_{(2n+1)}$ are Lagrange multipliers which transform like the $0-1$ and $0+1$ components of an $SO(1,1)$ vector under Lorentz transformations, $\partial_\pm = \partial_0 \pm \partial_1$, and our overall normalization for the action is chosen to reproduce the free fermion radius if ϕ is identified with $\phi + 2\pi$.

This action is invariant under the gauge transformation:

$$\delta\phi_{(n)} = \lambda_{(n-1)} - \lambda_{(n)}, \quad \delta A^\pm_{(n)} = \partial_\mp \lambda_{(n)} \tag{2.3}$$

(with $\lambda_{(-1)} \equiv 0$) which allows $\phi_{(n)}$ to be algebraically gauged away for $n \geq 1$. In this gauge, (2.2) simplifies to

$$\mathcal{S} = \frac{1}{4\pi}\int d^2x[\frac{1}{2}\partial_+\phi_{(0)}\partial_-\phi_{(0)} + A^-_{(0)}\partial_-\phi_{(0)} + \sum_{n=0}^{\infty}(A^-_{(2n)} - A^-_{(2n+2)})A^+_{(2n+1)}]. \tag{2.4}$$

The equations of motion for (2.4) are easily found to be

$$\partial_-\phi_{(0)} + A^+_{(1)} = 0, \quad \partial_-\partial_+\phi_{(0)} = \partial_- A^-_{(0)}\,, \tag{2.5}$$
$$A^+_{(1)} = A^+_{(3)} = A^+_{(5)} = \ldots\,, \quad A^-_{(0)} = A^-_{(2)} = A^-_{(4)} = \ldots\,.$$

If solutions to (2.5) are required to contain a finite number of non-vanishing fields, the solutions must satisfy

$$\partial_-\phi_{(0)} = 0, \quad 0 = A^+_{(1)} = A^+_{(3)} = \ldots, \quad 0 = A^-_{(0)} = A^-_{(2)} = \ldots\,, \tag{2.6}$$

which are the desired conditions.

Although the action of (2.4) might appear too trivial to have any useful applications, it is interesting to note that superstring field theory uses this type of action to describe Ramond-Ramond fields and their coupling to D-branes [10].

2.3 Discretized Chern-Simons

The action of (2.2) can be understood geometrically [13, 15] as a discretized version of the abelian Chern-Simons action in three dimensions. This is easily seen by writing (2.2) as

$$\begin{aligned}\mathcal{S} \;=\; & \frac{1}{4\pi}\sum_{n=0}^{\infty}\int d^2x[\frac{1}{2}(-1)^n\partial_+A^2_{(n)}\partial_-A^2_{(n)}(\delta x_2)^2 + (A^-_{(2n)} - A^-_{(2n+2)})A^+_{(2n+1)} + \\ & +A^-_{(2n)}(\partial^+A^2_{(2n)} + \partial^+A^2_{(2n+1)})\delta x_2 - A^+_{(2n+1)}(\partial^-A^2_{(2n)} + \partial^-A^2_{(2n+1)})\delta x_2],\end{aligned} \tag{2.7}$$

where $A^2_{(n)} = \phi_{(n)}/\delta x_2$ and δx_2 is a small positive constant. Now introduce a third coordinate x_2 which is discretized to take non-negative values $n\delta x_2$ and define $A^\mu(x_0, x_1, n\delta x_2) = A^\mu_{(n)}(x_0, x_1)$ for $\mu = 0, 1, 2$.

In the continuum limit as $\delta x_2 \to 0$, the first term of (2.7) drops out and the limit of the other terms become the abelian Chern-Simons action

$$\mathcal{S} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\int_0^\infty dx_2\int d^2x A_\mu\partial_\nu A_\rho = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\int d^3x A_\mu\partial_\nu A_\rho\,, \tag{2.8}$$

where the three-dimensional integration region is over the half-volume $x_2 \geq 0$. Note that the normalization factor of $\frac{1}{4\pi}$ implies that the Chern-Simons theory is at level $\frac{1}{2}$ [14]. (The normalization is fixed by requiring that ϕ is at the free fermion radius and that the gauge transformations of (2.3) are the discretized version of $\delta A_\mu = \partial_\mu \lambda$.)

In this discretized $D = 3$ form of the action, it is easy to explain the condition that solutions should contain only a finite number of non-vanishing fields. It is just the usual asymptotic condition that the fields should vanish at infinity.

The identification of (2.2) and (2.8) presents a puzzle since the continuum version of the Chern-Simons theory is not chiral, whereas (2.6) is. The chirality is hidden in the passage from discrete to continuous fields near the boundary. In the discrete version, A^+ is defined at $x_2 = (2n+1)\delta x_2$, but not at $x_2 = 2n\delta x_2$. To define A^+ at $x_2 = 2n\delta x_2$, one should take the average value of $A^+_{(2n-1)}$ and $A^+_{(2n+1)}$. Since $A^+_{(-1)}$ is undefined, this means fixing $A^+(x_0, x_1, 0) = A^+_{(1)}(x_0, x_1)$. So in the continuum version,

$$0 = A^+(x_0, x_1, \delta x_2) - A^+(x_0, x_1, 0) = \delta x_2 \partial_2 A^+(x_0, x_1, 0)\,.$$

But $\partial_2 A^+ = \partial^+ A_2$ on-shell, so

$$0 = \delta x_2 \partial^+ A_2(x_0, x_1, 0) = \partial^+ \phi_{(0)}\,,$$

which is the desired chirality condition, arising from a chiral boundary condition.

The $D = 3$ version of the MWYW action for a chiral scalar is easily generalized to a $D = 4p+3$ version of the MWYW action for a chiral $2p$-form. In this case, ϕ is a $2p$-form, A is a $2p+1$-form defined such that $A_{\mu_1 \ldots \mu_{2p+1}}$ are the Lagrange multipliers for $\mu_i = 0$ to $4p+1$,

$$A_{\mu_1 \ldots \mu_{2p} 4p+2} = \phi_{\mu_1 \ldots \mu_{2p}} / \delta x_{4p+2},$$

and the appropriate action is a discretized version of the $D = 4p+3$ Chern-Simons action:

$$\begin{aligned} \mathcal{S} &= \frac{1}{4\pi} \epsilon^{\mu_1 \ldots \mu_{2p+1} \nu \rho_1 \ldots \rho_{2p+1}} \int_0^\infty dx_{4p+2} \int d^{4p+2}x \, A_{\mu_1 \ldots \mu_{2p+1}} \partial_\nu A_{\rho_1 \ldots \rho_{2p+1}} \\ &= \frac{1}{4\pi} \epsilon^{\mu_1 \ldots \mu_{2p+1} \nu \rho_1 \ldots \rho_{2p+1}} \int d^{4p+3}x \, A_{\mu_1 \ldots \mu_{2p+1}} \partial_\nu A_{\rho_1 \ldots \rho_{2p+1}}\,, \end{aligned} \tag{2.9}$$

where the $(4p+3)$-dimensional integration is over the region $x_{4p+2} \geq 0$.

As discussed in [14], there is a problem with obtaining the partition function for a chiral $2p$-form from a path integral formalism. The problem is that the partition function depends on spin structure, but there is no such dependence, at least naively, in the action of (2.4). This problem is related to the fact that, when the chiral $2p$-form is coupled to a background $2p+1$-form gauge field $A_{\mu_1 \ldots \mu_{2p+1}}$, the partition function is not invariant under gauge transformations of $A_{\mu_1 \ldots \mu_{2p+1}}$. To define the partition function in [14], it was useful to introduce a level $\frac{1}{2}$ Chern-Simons action for $A_{\mu_1 \ldots \mu_{2p+1}}$, which was precisely the action in (2.9). This suggests that the Chern-Simons form of the MWYW action might be useful for resolving the problem.

3 Manifestly covariant actions for $D = 4$ self-dual Yang-Mills

3.1 MWYW action for self-dual Yang-Mills

Self-dual Yang-Mills is described by a gauge field A^I_μ whose field-strength $F^I_{\mu\nu} = \partial_{[\mu} A^I_{\nu]} + f^I_{JK} A^J_\mu A^K_\nu$ is self-dual, i.e.

$$F^I_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma\, I}\,, \tag{3.1}$$

where the spacetime signature is $(2, 2)$. Although (3.1) can be obtained as an equation of motion from the actions of [16] or [17], these actions are not SO(2,2) Lorentz-invariant and require the four-dimensional spacetime to be Kahler.

By replacing the chiral boson constraint with (3.1), it is straightforward to use the MWYW method to construct a manifestly Lorentz-invariant action for self-dual Yang-Mills. The analogue of (2.4) is

$$\mathcal{S} = \int d^4x Tr[\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + G^{\mu\nu}_{(0)}(F_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}) + \sum_{n=0}^{\infty} (-1)^n G^{\mu\nu}_{(n)} G_{(n+1)\mu\nu}]\,, \tag{3.2}$$

where $G^I_{(n)\mu\nu}$ are Lagrange multipliers which will be taken to be anti-self-dual, i.e. $G^I_{(n)\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma\, I}_{(n)}$. Unlike the actions of [16] and [17], (3.2) can be generalized to any four-dimensional background. Note that the first terms in the action are the same as the action of [18], and the infinite sum removes the propagating field in $G^{\mu\nu\, I}_{(0)}$.

The equations of motion for (3.2) are easily found to be

$$(\partial^\mu \delta^I_J + f^I_{JK} A^{\mu\, K})[F^J_{\mu\nu} + 2G^J_{(0)\mu\nu}] = 0, \quad F^I_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma\, I} - G^{\mu\nu\, I}_{(1)}\,, \tag{3.3}$$
$$G^{\mu\nu\, I}_{(0)} = G^{\mu\nu\, I}_{(2)} = \ldots\,, \quad G^{\mu\nu\, I}_{(1)} = G^{\mu\nu\, I}_{(3)} = \ldots\,.$$

If the solutions to (3.3) are required to contain a finite number of non-vanishing fields, the solutions must satisfy

$$F^I_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma\, I}, \quad G^{\mu\nu\, I}_{(n)} = 0 \tag{3.4}$$

which are the desired conditions.

3.2 Self-dual Maxwell theory from discrete form of a 5D action

The analogue of (2.4) for self-dual Yang-Mills is (3.2), and just as (2.4) can be obtained by gauge-fixing the action (2.2), (3.2) can be obtained by gauge-fixing an action, at least in the abelian case. The action (3.2) can be written in form notation as

$$\mathcal{S} = \int [\frac{1}{2} F\wedge * F + \frac{1}{2} G_{(0)}\wedge(*F - F) + \sum_{n=0}^{\infty}(-1)^n G_{(n)}\wedge * G_{(n+1)}]\,. \tag{3.5}$$

In the abelian case, this can be obtained by gauge-fixing the following action:

$$\begin{aligned}\mathcal{S} \;=\; &\int \sum_{n=0}^{\infty}[\frac{1}{2}(-1)^n dB_{(n)}\wedge * dB_{(n)} + (G_{(2n)} - G_{(2n+2)})\wedge * G_{(2n+1)} + \\ &+G_{(2n)}\wedge * d(B_{(2n)} + B_{(2n+1)}) - G_{(2n+1)}\wedge * d(B_{(2n+1)} + B_{(2n+2)})\,,\end{aligned} \tag{3.6}$$

where the $B_{(n)}$ are 1-forms, with $B_{(0)} = A$, $dB_{(0)} = F$. (3.6) is invariant under the symmetries:

$$\delta B_{(n)} = \lambda_{(n-1)} - \lambda_{(n)}\,, \quad \delta G_{(n)} = \frac{1}{2}[d\lambda_{(n)} - *d\lambda_{(n)}]\,, \tag{3.7}$$

(with $\lambda_{(-1)} \equiv 0$) where the parameters $\lambda_{(n)}$ are now 1-forms. This allows $B_{(n)}$ to be algebraically gauged away for $n \geq 1$, which produces the action (3.2), where $\frac{1}{2}G_{(0)}\wedge(*F - F) = G_{(0)}\wedge * F$ since $G_{(0)}$ is anti-self-dual.

The chiral boson action was seen to be related to a 3-dimensional Chern-Simons theory, and the formal similarity with that case suggests seeking a 5-dimensional theory whose discretisation gives an action of this type. A simple guess would be to define five-dimensional fields

$$C^-(x^\mu, 2n\delta x^4) = G_{(2n)}(x^\mu), \quad C^+(x^\mu, (2n+1)\delta x^4) = G_{(2n+1)}(x^\mu)\,,$$

$$C(x^\mu, n\delta x^4) = B_{(n)}(x^\mu)/\delta x^4\,.$$

In the continuum limit $\delta x^4 \to 0$, (3.6) becomes

$$\mathcal{S} = \epsilon^{\mu\nu\rho\sigma} \int d^5x [C^+_{\mu\nu}\partial_4 C^-_{\rho\sigma} + C^-_{\mu\nu}\partial_\rho C_\sigma - C^+_{\mu\nu}\partial_\rho C_\sigma]\,. \tag{3.8}$$

However, the anti-self-duality of $C^\pm_{\mu\nu}$ is not a covariant constraint in five dimensions, so (3.8) is not SO(3,2) Lorentz-covariant.

To get a Lorentz-covariant action, one could try writing an alternative action involving infinite fields in which the multiplier fields $G_{(n)}$ are not restricted to be anti-self-dual. For example, the action

$$\begin{aligned}\mathcal{S} \;&=\; \int \sum_{n=0}^{\infty} \Big[\frac{1}{2}(-1)^n dB_{(n)}\wedge * dB_{(n)} + (G_{(2n)} - G_{(2n+2)})\wedge * G_{(2n+1)} + \\ &\quad + G_{(2n)}\wedge * d(B_{(2n)} + B_{(2n+1)}) - G_{(2n+1)}\wedge * d(B_{(2n+1)} + B_{(2n+2)})\Big]\end{aligned} \tag{3.9}$$

is invariant under

$$\delta B_{(n)} = \lambda_{(n-1)} - \lambda_{(n)}, \quad \delta G_{(n)} = d\lambda_{(n)}\,, \tag{3.10}$$

for general 2-forms $G_{(n)}$. But on gauge-fixing, it gives $G_{(0)}\wedge * F$ instead of $G_{(0)}\wedge(*F - F)$. One way of obtaining $G_{(0)}\wedge(*F - F)$ would be to add to (3.9) the term

$$\mathcal{S}' = \int \sum_{n=0}^{\infty} \frac{1}{2} G_{(0)}\wedge dB_{(n)} \tag{3.11}$$

(which is gauge-invariant up to a surface term), but (3.11) is non-local in the fifth dimension. Note that the simplest guess for a covariant five-dimensional limit of (3.9) would be $\int CdC$, but such an action for a 2-form C is trivial since the integrand is a total derivative.

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4 Manifestly supersymmetric actions for $D = 10$ super-Yang-Mills

4.1 MWYW action for $D = 10$ super-Yang-Mills

The physical fields of ten-dimensional super-Yang-Mills consist of a gauge field A_μ^I and a Majorana-Weyl spinor field Ψ_α^I where I takes values in the adjoint representation. On-shell, these fields satisfy the equations of motion

$$\mathcal{D}_\mu F^{\mu\nu} = 0, \quad \Gamma_\mu^{\alpha\beta}\mathcal{D}^\mu\Psi_\alpha = 0\,, \tag{4.1}$$

which can be obtained from the action

$$\int d^{10}x Tr(\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + i\Psi_\alpha\Gamma_\mu^{\alpha\beta}\mathcal{D}^\mu\Psi_\beta)\,, \tag{4.2}$$

where $\mathcal{D}^\mu\Psi_\beta = \partial^\mu\Psi_\beta + [A^\mu, \Psi_\beta]$ and $\Gamma_\mu^{\alpha\beta}$ are the symmetric 16×16 Pauli matrices in ten dimensions. Although this action is invariant under the global supersymmetry transformations

$$\delta A_\mu^I = i\epsilon_\alpha\Gamma_\mu^{\alpha\beta}\Psi_\beta, \quad \delta\Psi_\alpha = -\frac{1}{2}F^{\mu\nu}(\Gamma_{\mu\nu})_\alpha^\beta\epsilon_\beta\,, \tag{4.3}$$

this invariance is not manifest.

It is well-known [19] that the equations of motion of (4.1) can be written in manifestly supersymmetric notation as

$$D^\alpha\Gamma_{\alpha\beta}^{\mu_1\ldots\mu_5}\mathcal{A}^\beta(x,\theta) = 0\,, \tag{4.4}$$

where θ^α is a Majorana-Weyl spinor variable, $\Gamma^{\mu_1\ldots\mu_5}$ is the anti-symmetrized product of five Γ-matrices, $\mathcal{A}^\alpha$ is a spinor superfield whose component expansion is

$$\mathcal{A}^\alpha = \xi^\alpha + \Gamma_\mu^{\alpha\beta}\theta_\beta A^\mu + \Gamma^{\mu\,\alpha\beta}\theta_\beta(\theta_\gamma\Gamma_\mu^{\gamma\delta}\Psi_\delta) + \ldots \tag{4.5}$$

and

$$D^\alpha\Gamma_{\alpha\beta}^{\mu_1\ldots\mu_5}\mathcal{A}^\beta = \Gamma_{\alpha\beta}^{\mu_1\ldots\mu_5}\left[\left(\frac{\partial}{\partial\theta_\alpha} + i\theta_\gamma\Gamma_\mu^{\alpha\gamma}\right)\mathcal{A}^\beta + \{\mathcal{A}^\alpha, \mathcal{A}^\beta\}\right].$$

Note that (4.4) is invariant under $\delta\mathcal{A}^\alpha = D^\alpha\Lambda$ for an arbitrary superfield Λ, which allows the component field ξ^α to be gauged away. In this gauge, (4.4) implies that the higher components in the θ expansion of $\mathcal{A}_\alpha$ are all related to ψ_β and A^μ.

The natural generalization of (2.4) and (3.2) for this case is the action

$$\mathcal{S} = \int d^{10}x \int d^{16}\theta\; Tr[G_{(0)\,\mu_1\ldots\mu_5}D^\alpha\Gamma_{\alpha\beta}^{\mu_1\ldots\mu_5}\mathcal{A}^\beta + \sum_{n=0}^{\infty}(-1)^n G_{(n)}^{\mu_1\ldots\mu_5}G_{(n+1)\mu_1\ldots\mu_5}]\,, \tag{4.6}$$

where $G_{(n)}^{\mu_1\ldots\mu_5}$ is an unconstrained superfield which takes values in the adjoint representation. Note that one could also include a quadratic term for $\mathcal{A}^\alpha$ of the type $(D^\alpha\Gamma_{\alpha\beta}^{\mu_1\ldots\mu_5}\mathcal{A}^\beta)(D^\gamma\Gamma_{\mu_1\ldots\mu_5\,\gamma\delta}\mathcal{A}^\delta)$, but such a term vanishes in ten dimensions.[1]

[1] We would like to thank Mario Tonin for pointing this out to us.

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The equations of motion from varying the superfields in (4.6) are

$$D^\alpha \Gamma^{\mu_1 \ldots \mu_5}_{\alpha\beta} G_{(0)\mu_1\ldots\mu_5} = 0, \quad D^\alpha \Gamma^{\mu_1\ldots\mu_5}_{\alpha\beta} \mathcal{A}^\beta + G^{\mu_1\ldots\mu_5}_{(1)} = 0\,, \tag{4.7}$$

$$G^{\mu_1\ldots\mu_5}_{(0)} = G^{\mu_1\ldots\mu_5}_{(2)} = \ldots\,, \quad G^{\mu_1\ldots\mu_5}_{(1)} = G^{\mu_1\ldots\mu_5}_{(3)} = \ldots\,.$$

If solutions to (4.7) are required to contain a finite number of non-vanishing fields, the solutions must satisfy

$$D^\alpha \Gamma^{\mu_1\ldots\mu_5}_{\alpha\beta} \mathcal{A}^\beta = 0, \quad G^{\mu_1\ldots\mu_5}_{(n)} = 0\,, \tag{4.8}$$

which are the desired conditions.

4.2 Super-Maxwell theory from discrete form of an 11D action

Just like the actions of (2.4) and (3.2) , the action of (4.6) in the abelian case can be obtained by gauge-fixing the following action:

$$\begin{aligned} \mathcal{S} \;=\; & \int d^{10}x \int d^{16}\theta \sum_{n=0}^{\infty} \Big[(G_{(2n)} - G_{(2n+2)}) \wedge * G_{(2n+1)} + \\ & + G_{(2n)} \wedge (*H_{(2n)} + *H_{(2n+1)}) - G_{(2n+1)} \wedge (*H_{(2n+1)} + *H_{(2n+2)}) \Big]\,, \end{aligned} \tag{4.9}$$

where $H^{\mu_1\ldots\mu_5}_{(n)} = D^\alpha \Gamma^{\mu_1\ldots\mu_5}_{\alpha\beta} \mathcal{A}^\beta_{(n)}$, $\mathcal{A}^\beta_{(0)}$ is the original gauge superfield, and we have switched to form notation. (Note that $H_{(n)} \wedge *H_{(n)}$ vanishes identically in ten dimensions.)

This action is invariant under the gauge transformations

$$\delta \mathcal{A}^\alpha_{(n)} = \lambda^\alpha_{(n-1)} - \lambda^\alpha_{(n)}, \quad \delta G^{\mu_1\ldots\mu_5}_{(n)} = D^\alpha \Gamma^{\mu_1\ldots\mu_5}_{\alpha\beta} \lambda^\beta_{(n)}\,, \tag{4.10}$$

which can be used to algebraically gauge-fix $\mathcal{A}^\alpha_{(n)} = 0$ for $n \geq 1$, which returns (4.9) to the action of (4.6).

Defining

$$C^-(x^\nu, 2n\delta x^{10}) = G_{(2n)}, \quad C^+(x^\nu, (2n+1)\delta x^{10}) = G_{(2n+1)}(x^\nu)\,,$$

$$\mathcal{C}^\alpha(x^\nu, n\delta x^{10}) = \mathcal{A}^\alpha_{(n)}(x^\nu)/\delta x^4,$$

one can write (4.9) in the limit $\delta x^{10} \to 0$ as the eleven-dimensional action

$$\mathcal{S} = \int d^{11}x [C^+_{\mu_1\ldots\mu_5} \partial_{10} C^{-\,\mu_1\ldots\mu_5} + (C^-_{\mu_1\ldots\mu_5} - C^+_{\mu_1\ldots\mu_5}) D^\alpha \Gamma^{\mu_1\ldots\mu_5}_{\alpha\beta} \mathcal{C}^\beta]\,. \tag{4.11}$$

Of course, this action is not SO(10,1) invariant since μ ranges from 0 to 9 and α ranges from 1 to 16.

5 Comments on infinite fields

In this paper, we constructed manifestly covariant actions for $D = 4$ self-dual Yang-Mills and $D = 10$ super-Yang-Mills theories using an infinite number of fields. Although actions involving an infinite number of fields are not often used, there are several instances where they naturally arise.

One instance was already mentioned and involves discretizing a spacetime dimension. It would be interesting to try to generalize the actions of (2.2), (3.6) and (4.9) to the non-abelian case.

A second instance of actions involving infinite fields is that of a Kaluza-Klein reduction, in which a Fourier expansion in the coordinate of a compact dimension leads to an infinite number of fields.

A third instance of actions involving infinite fields are the harmonic superspace actions of [20]. The fields in these actions depend polynomially on a bosonic "harmonic" variable u, and Taylor expanding in u produces the infinite fields. In the superstring field theory action for Ramond-Ramond fields, which is of the MWYW type, the infinite fields arise in precisely this manner where the harmonic variable u is constructed from a combination of the bosonic ghost zero modes [10]. Perhaps the infinite fields in the actions of (3.2) and (2.4) can also be interpreted as a Taylor expansion in some harmonic variable. This would connect with earlier attempts to construct actions for $D = 4$ self-dual Yang-Mills and $D = 10$ super-Yang-Mills using harmonic variables [21, 22].

Acknowledgments

NB would like to thank N. Nekrassov for discussions on discretized Chern-Simons, and Queen Mary and Westfield College for their hospitality. The work of NB was partially supported by CNPq grant number 300256/94.

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